Statistical analysis for the Johnson-Mehl germination-growth model

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- Estimating parameters of specific parametric models for the conditional intensity by maximum likelihood.
- Model checking by new functional summary statistics related to the inhomogeneous K- function and to the Palm distribution of the typical Johnson-Mehl cell.

Outline of Topics

Definition

- $\Phi \equiv \{(x_i, t_i)\} \subset \mathbb{R}^d \times [0, \infty)$: Primary process, a space-time Poisson process with intensity function $\kappa(t)$.

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- Time $T((x,t),y)$

$$T((x,t),y) = t + \|x - y\|/v, \quad (x,t) \in \mathbb{R}^d \times [0,\infty) \text{ and } y \in \mathbb{R}^d.$$

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- New points and cells form and grow only in uncovered space.

Definition(cont.)

- Growth ceases for each cell whenever and wherever it touches a neighbouring cell.

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- Cells: $C_i = C(x_i, t_i) = \{y \in \mathbb{R}^d : T_i(y) \leq T_j(y)$ for all $j \neq i$ with $(x_j, t_j) \in \Psi\}$, where $T_j(x_i) = T((x_j, t_j), x_i)$.

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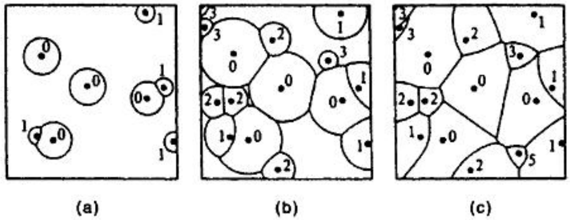
- 

Figure: The Johnson-Mehl model for times (a) t=1, (b) t=3, and (c) t=7

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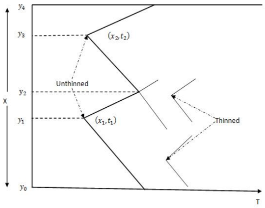

Figure: Thinned and unthinned points of a germination-growth process

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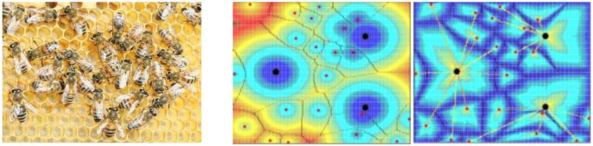

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Figure: Voronoi tessellation; cells are convex polyhedra.
J-M tessellation: cells here are non-convex sets with curved boundaries

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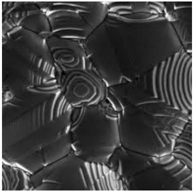
- 

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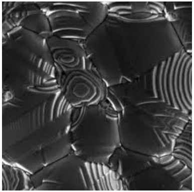
- 

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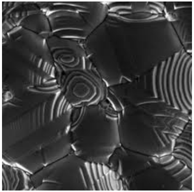
- 

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- The statistical aspects?

First-order properties

- The intensity of Ψ is given by

$$\rho(t) = \exp\left(- \int \int_{C(0,t)} \kappa(s) \, \mathrm{d}x \, \mathrm{d}s \right) \kappa(t)$$

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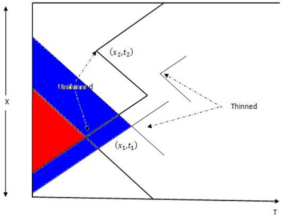
- 

Figure: Cone generated by (x_1, t_1) (Red colored area) and by the thinned point (Blue+Red colored area)

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- By (1) we have obtained a second-order non-linear differential equation with solution

$$\kappa(t) = \frac{c_1}{\cos^2(c_2 t + c_3)}, \tag{2}$$

c_1, c_2 and c_3 are constants (and $c_2 t + c_3 \neq k\pi/2,\ k \in \mathbb{Z}\backslash\{0\}$).

Second-order properties

- Let $(x, s) \neq (y, t)$ in $\mathbb{R}^d \times [0, \infty)$ with distance $r = \|x - y\|$, (x, s) and (y, t) are in Ψ, if $T((x, s), y) > t$ and $T((y, t), x) > s$, or $r > v|s - t|$.

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$$\rho_0^{(2)}(r,s,t) = \kappa(s)\kappa(t)\mathbf{1}[r > v|s-t|] \exp\left(-\int_0^{\max\{s,t\}} \kappa(u) V_{\cup}(r, s-u, t-u)\,\mathrm{d}u\right).$$

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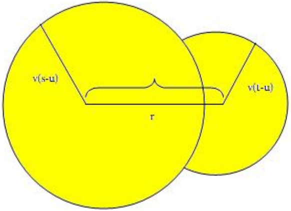

Figure: $V_\cup(r, s-u, t-u)$

Second-order properties

- The pair correlation function is given by

$$g(r,s,t) = \mathbf{1}[v|s-t| < r < v(s+t)] \exp\left(-\int_0^{(s+t-r/v)/2} \kappa(u) V_\cap(r, s-u, t-u)\, \mathrm{d}u \right) + \mathbf{1}[r \geq v(s+t)].$$

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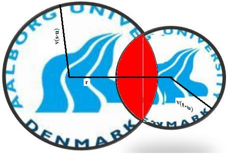

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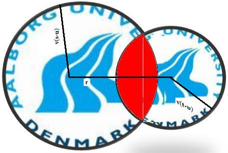

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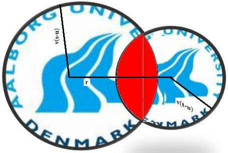

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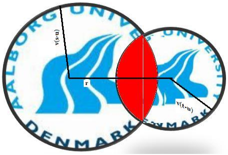

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- $V_\cap > 0 \iff u < (s+t-r/v)/2$ when $r > v|s-t|$.
- Since $g(r,s,t)$ is not a function of r and $s-t$ only. Therefore, Ψ is not second-order intensity-reweighted stationary.

$V_\cap(r, s-u, t-u)$

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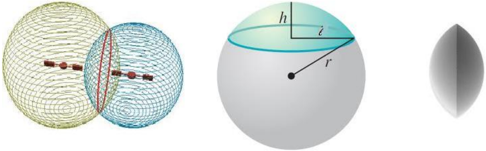

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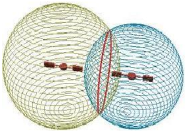
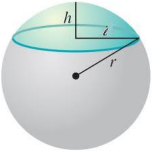

- volume of a d-dimensional hyper-spherical cap:

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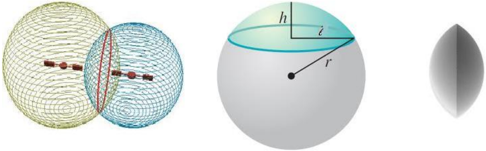

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$$V_\cap(r, s-u, t-u) = V_d\left(v(s-u), \frac{[v(t-u)]^2 - (r - v(s-u))^2}{2r}\right) + V_d\left(v(t-u), \frac{[v(s-u)]^2 - (r - v(t-u))^2}{2r}\right).$$

Inhomogeneous K-function-like summary statistics and their Non-parametric Estimation

- For $R > 0$, define

$$K_1(R) = \mathrm{E} \sum_{i \neq j} \frac{\mathbf{1}\left[x_i \in W,\ v(t_i + t_j) < \|x_i - x_j\| \leq R\right]}{|W|\rho(t_i)\rho(t_j)}$$

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- An unbiased estimator of $K_1(R)$:

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Summary statistics based on the characteristics for the typical Johnson-Mehl cell

- The Palm distribution of the typical cell $\mathscr{C}$ is defined by

$$\zeta|W|P(\mathscr{C} \in F) = \mathrm{E}\sum_i \mathbf{1}\left[T_j(x_i) > t_i \ \forall j \neq i, x_i \in W,\ C_i - x_i \in F\right]$$

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- Let o denote the origin.

$$C(o,t|\Phi) = \{y \in \mathbb{R}^d : T((o,t),y) \leq T((x_j,t_j),y) \text{ for all } (x_j,t_j) \in \Psi\}$$

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- Hence, by Slivnyak-Mecke formula

$$P(\mathscr{C} \in F) = \int P\left(T_j(o) > t \ \forall j,\ C((o,t)|\Phi) \in F\right) \kappa(t)\, \mathrm{d}t/\zeta.$$

Palm distribution of the typical shortest nucleus-boundary distance R

- In the Johnson-Mehl case, the distribution function for R is

$$D(r) = P(R \le r) = 1 - \iint P\left(\Phi \cap H(t,r) = \emptyset\right) \kappa(t)\,\mathrm{d}x\,\mathrm{d}t/\zeta, \quad r > 0,$$

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$$H(t,r) = \{(y,u) \in \mathbb{R}^d \times [0,t] : \|y\| \le 2r + v(t-u)\} \\ \cup \{(y,u) \in \mathbb{R}^d \times (t, t + r/v] : v(t-u) \le \|y\| \le 2r + v(t-u)\}.$$

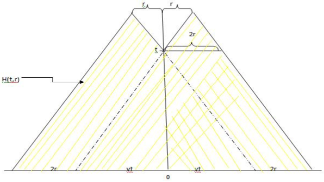

Figure: Example of the region $H(t,r)$.

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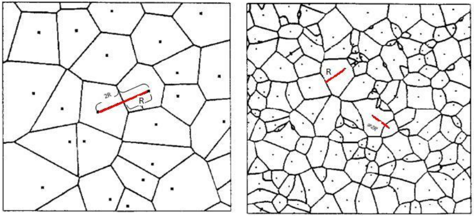
- 

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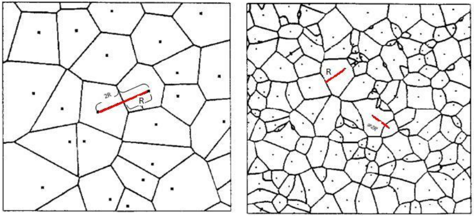
- 

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- By ignoring edge effects, a ratio unbiased non-parametric estimate of $D(r)$ is

$$\hat{D}(r) = \frac{1}{|W|} \sum_i \frac{\mathbf{1}\left[T_j(x_i) > t_i \ \forall j \neq i, x_i \in W, \ R_i \leq r\right]}{\hat{\zeta}}.$$

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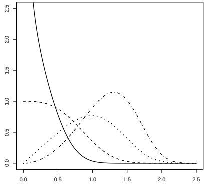

Figure: Behavior of $\rho(t)$ for model M1 with $d = 2$ and $\alpha = v = 1$, when $\beta = 0.5$ (solid line), $\beta = 1$ (dashed line), $\beta = 2$ (dotted line), and $\beta = 3$ (dot-dashed line).

Pair correlation function under model M1

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$$g(r,s,t) = 1[r > v|s-t|] \exp\left(\frac{\alpha(v(\beta+1)+\beta)}{\beta(\beta+1)2^\beta} (s+t-r/v)^{\beta+1} \right)$$

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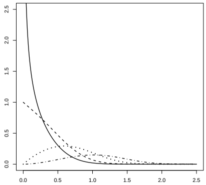

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Likelihood when Φ is defined on $W \times [0, \infty)$

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$$\lambda(x, t|\mathcal{H}_t)\,\mathrm{d}t = \mathbf{1}[T_i(x) > t\ \forall\, t_i < t\ \text{ with } (x_i, t_i) \in \Psi_W]\,\mathcal{K}(\mathrm{d}t), \quad (x, t) \in W \times [0, \infty),$$

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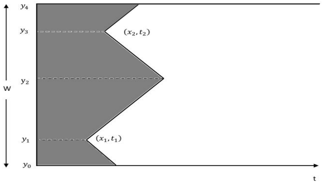
- 

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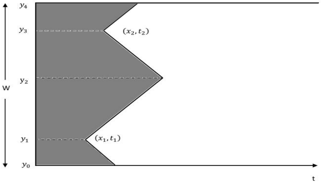
- 

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Release of neurotransmitter at the neuromuscular junction

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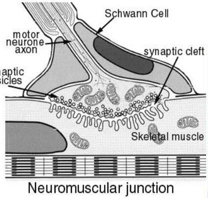

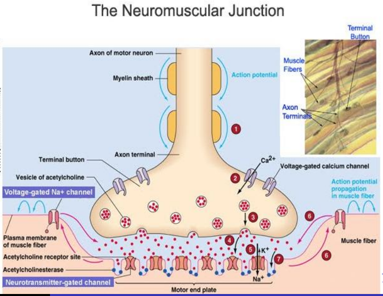

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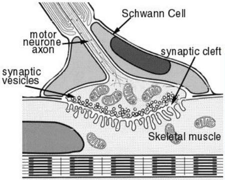

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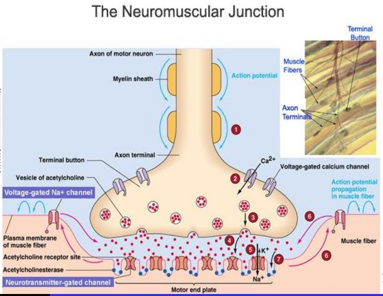

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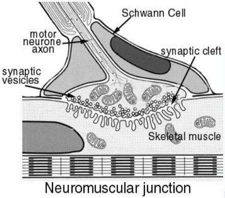

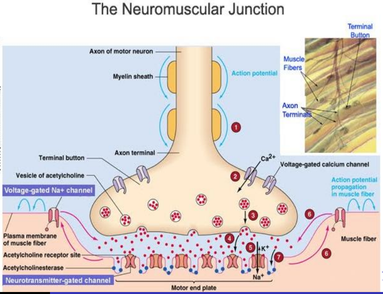

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- Finally, 746 experiments with 101 experiments with no germinated seed and 645 with at least one germinated seed are obtained. The frequencies of 1's, ..., 4's now being 387, 210, 45, 3, respectively.

Model checking

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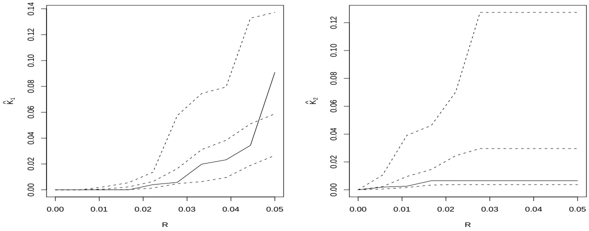

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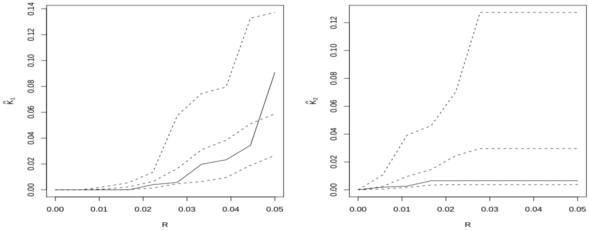

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Work in progress

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Thank You!