Continuum Percolation in the β skeleton graph

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Outline

Continuum percolation result in β skeleton graph for Poisson stationary point process with unit intensity in $\mathbb{R}^2$.

Some applications

- Ferromagnetism (at low temperature) and Ising model
- Disordered electrical networks (electrical resistance of a mixture of two materials)
- Cancerology for the study of the growth of tumor when the cancer cells suddently begin to invade healthy tissue.
- Epidemics and fires in orchards

Bibliography

- Meester and Roy [5] for continuum percolation
- Häggström and Meester [4] proposed results for continuum percolation problems for the k-nearest neighbor graph under Poisson process
- Bertin et al. [2] proved the result for the Gabriel graph
- Bollobás and Riordan [3] critical probability for random Voronoi percolation in the plane is 1/2.
- Balister and Bollobás [1] gave bounds on k for the k-nearest neighbor graph for percolation

Graphs $G_\beta = (V, E, N_\beta)$

$$(u,v) \in E \Leftrightarrow L_{u,v}(\beta) \cap V = \emptyset \text{ respectively } C_{u,v}(\beta) \cap V = \emptyset.$$

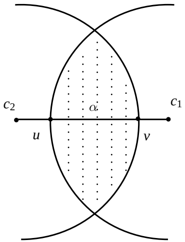
$$L_{u,v}(\beta) = D\left(c_1 = u + \frac{\beta(\alpha)}{2}(v-u), \alpha\frac{\beta(\alpha)}{2}\right)$$

$$\cap\, D\left(c_2 = v + (u-v)\frac{\beta(\alpha)}{2}, \alpha\frac{\beta(\alpha)}{2}\right)$$

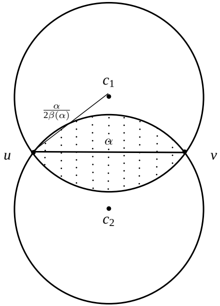
$$C_{u,v}(\beta) = D\left(c_1, \alpha\frac{\beta(\alpha)}{2}\right) \cup D\left(c_2, \alpha\frac{\beta(\alpha)}{2}\right)$$

with $\delta(c_1,u) = \delta(c_1,v) = \delta(c_2,u) = \delta(c_2,v) = \alpha\frac{\beta(\alpha)}{2}$ and $\beta(\alpha) \geq 1$.
For $0 < \beta(\alpha) \leq 1$:

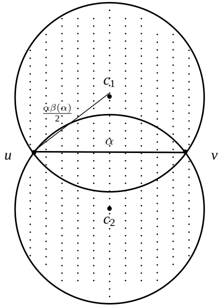
$$C_{u,v}(\beta) = D\left(c_1, \frac{\alpha}{2\beta(\alpha)}\right) \cap D\left(c_2, \frac{\alpha}{2\beta(\alpha)}\right)$$



$L_{u,v}(\beta)$ with $\beta \geq 1$



$C_{u,v}(\beta)$ with $\beta < 1$



$C_{u,v}(\beta)$ with $\beta > 1$

1-independent percolation

To prove that continuous percolation occurs, we shall compare the process to various bond percolation models on $\mathbb{Z}^2$. In these models, the states of the edges are not be independent.

Definition

A bond percolation model is 1-independent if whenever E_1 and E_2 are sets of edges at graph distance at least 1 from each another (i.e., if no edge of E_1 is incident to an edge of E_2) then the state of the edges in E_1 is independent of the state of the edges in E_2.

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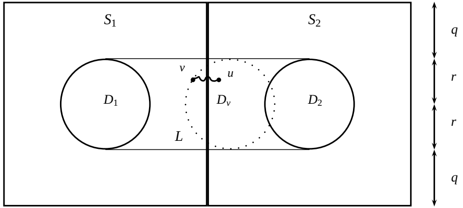
The Rolling Ball Method



Comparison with $\mathbb{Z}^2$

- Write $u \sim v$ if uv is an edge of the underlying graph
- Percolation = infinite path : a sequence $u_1, u_2 \ldots$ with $u_i \sim u_{i+1}$ for all i.
- Let $\mathcal{E}_{S_1,S_2}$ be the event that every vertex u_1 in the central disk C_1 of S_1 is joined to at least one vertex v in the central disk C_2 of S_2 by a $G_\beta -$ path, regardless of the state of the Poisson process outside of S_1 and S_2.
- Each vertex $(i,j) \in \mathbb{Z}^2$ corresponds to a square $[Ri, R(i+1)] \times [Rj, R(j+1)] \in \mathbb{R}^2$, where $R = 2r + 2q$, and an edge is open between adjacent vertices (corresponding to squares S_1 and S_2) if both events $\mathcal{E}_{S_1,S_2}$ and $\mathcal{E}_{S_2,S_1}$ hold.
- 1-independent model on $\mathbb{Z}^2$ since the event $\mathcal{E}_{S_1,S_2}$ depends only on the Poisson process within the region S_1 and S_2.

Comparison with $\mathbb{Z}^2$

- Any open path in $\mathbb{Z}^2$ corresponds to a sequence of events $\mathcal{E}_{S_1,S_2}, \mathcal{E}_{S_2,S_3} \ldots$ that occur, where S_i is the square associated with a site in $\mathbb{Z}^2$.
- Every vertex u_1 of the original Poisson process that lies in the central disk C_1 of S_1 now has an infinite path leading away from it, since one can find points u_i in the central disk of S_i and paths from u_{i-1} to u_i inductively for every $i \geq 1$.
- One can choose r, q and β so that the probability that the intersection of these events is large and then we will apply the theorem of Balister, Bollobas and Walters.

A result of a 1-independent bond percolation on $\mathbb{Z}^2$

Theorem (Balister, Bollobas, Walters. *Random Structures and Algorithms*, 2005)

If every edge in a 1-independent bond percolation model on $\mathbb{Z}^2$ is open with probability at least 0.8639*, then almost surely there is an infinite open component. Moreover, for any bounded region, there is almost surely a cycle of open edges surrounding this region.*

The main result

Let E_{S_1,S_2} be the event that for every point $v \in C_1 \cup L$, there is a u such that :
a) $v \sim u$;
b) $d(u,v) \leq s$; and
c) $u \in D_v$, where D_v is the disk of radius r inside $C_1 \cup L \cup C_2$ with v on its C_1-side boundary (the dotted disk in Figure 1).
If E_{S_1,S_2} holds, then every vertex v in C_1 must be joined by a $G_\beta-$path to a vertex in C_2, since each vertex in $C_1 \cup L$ is joined to a vertex whose disk D_v is further along in $C_1 \cup L \cup C_2$.

The main result

$$E_{S_1,S_2} = \{\varphi \in \Omega / \forall v \in \varphi_{C_1 \cup L},\ \exists u \in \varphi_{D_v \cap D(v,s)},\ (\varphi - \delta_v - \delta_u)(N_\beta(uv)) = 0\}$$

$$A_1 = \{\varphi \in \Omega / \varphi(D_0) > 0\}$$

$$A = E_{S_1,S_2} \cap E_{S_2,S_1} \cap A_1$$

Theorem

We can find s, r and β, function of the length of edges, so that $p(\bar{A}) \leq 0.1361$.

$$\bar{E}_{S_1,S_2} \cup \bar{A}_1 \subset \bar{A}_1 \cup A_2 \cup A_3$$

$$A_2 = \{\varphi \in \Omega / \exists v \in \varphi_{C_1 \cup L},\ (\varphi - \delta_v)(D_v \cap D(v,s)) = 0\}.$$
$$A_3 = \{\varphi \in \Omega / \exists v \in \varphi_{C_1 \cup L},\ \forall u \in \varphi_{D_v \cap D(v,s)},\ (\varphi - \delta_v - \delta_u)(N_\beta(uv)) > 0\}.$$

$P(\bar{A}_1) = e^{-\pi r^2}$. Using Campbell's theorem and Slyvnyak's theorem :
Given $A_{D_v} = \{\varphi \in \Omega / \varphi(D_v \cap D(v,s)) = 0\}$ and
$A_{D_0} = \{\varphi \in \Omega / \varphi(D_O \cap D(O,s)) = 0\}$, it comes

$$\mathbb{1}_{A_2}(\varphi) \leq \sum_{v \in \varphi} \mathbb{1}_{[C_1 \cup L]}(v) \mathbb{1}_{A_{D_v}}(\varphi - \delta_v).$$

$$P(A_2) \leq |C_1 \cup L|\, P^!_O(A_{D_0}) = |C_1 \cup L| P(A_{D_0}) = 2r(2r+2s)e^{-|D_O \cap D(O,s)|}$$

For the last probability, by introducing the following events

$$A_v = \{\varphi \in \Omega / \forall u \in \varphi_{D_v \cap D(v,s)},\ (\varphi - \delta_u)(N_\beta(uv)) > 0\}$$
$$A_O = \{\varphi \in \Omega / \forall u \in \varphi_{D_O \cap D(O,s)},\ (\varphi - \delta_u)(N_\beta(uO)) > 0\}$$
$$A_{Ou} = \{\varphi \in \Omega / \varphi(N_\beta(Ou)) > 0\}.$$
$$1_{A_3}(\varphi) = \max_{v \in \varphi} 1_{[C_1 \cup L]}(v) 1_{A_v}(\varphi - \delta_v) \leq \sum_{v \in \varphi} 1_{[C_1 \cup L]}(v) 1_{A_v}(\varphi - \delta_v).$$
$$P(A_3) \leq |C_1 \cup L|\, P^!_O(A_0) = |C_1 \cup L|\, P(A_O).$$
$$1_{A_O}(\varphi) \leq \sum_{u \in \varphi} 1_{D_O \cap D(O,s)}(u) 1_{A_{Ou}}(\varphi - \delta_u),$$
$$P(A_O) \leq \int_{D_O \cap D(O,s)} P^!_u(A_{Ou}) du = \int_{D_O \cap D(O,s)} (1 - e^{-|N_\beta(Ou)|}) du.$$
$$P(A_3) \leq |C_1 \cup L| \int_{D_O \cap D(O,s)} (1 - e^{-|N_\beta(Ou)|}) du.$$

Lemma

$$P(\bar{E}_{S_1,S_2} \cup \bar{A}_1) \leq e^{-\pi r^2} + 2r(2r+2q)e^{-|D_O \cap D(O,s)|}$$
$$+ 4r(2r+2q)\int_0^s \alpha \arccos\left(\frac{\alpha}{2r}\right)\left(1 - e^{-|N_\beta(\alpha)|}\right) d\alpha.$$

Remark : we choose the best q so that every neighborhood of two differents points inside $C_1 \cup L$ stay inside the rectangular zone $S_1 \cup S_2$. We are looking for a function β constant on an interval $[0, t]$ and function of α on the interval $[t, s]$ so that $|N_\beta(\alpha)| = |N_\beta(t)|$ for all α in $[t, s]$. We have :

$$P(\bar{E}_{S_1,S_2} \cup \bar{A}_1) \leq e^{-\pi r^2} + 2r(2r+2q)e^{-|D_O \cap D(O,s)|}$$

$$+4r(2r+2q)\int_0^t \alpha \arccos\left(\frac{\alpha}{2r}\right)\left(1 - e^{-|N_\beta(\alpha)|}\right) d\alpha$$

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Numerical results

β	N_β	r	s	$a\ (t = a/100 \times s)$
1 (Gabriel Graph)	$L_{u,v}(1)$	1.437	2.625	1.025
2 (RNG Graph)	$L_{u,v}(2)$	1.491	2.731	0.631
3	$L_{u,v}(3)$	1.515	2.824	0.484
2	$C_{u,v}(2)$	1.6	2.882	0.176
3	$C_{u,v}(3)$	1.7	2.862	0.087
$1/2$	$C_{u,v}(1/2)$	1.4	2.522	2.71
$0 < \beta \leq 0.001$	$C_{u,v}(\beta)$	1.31	2.6	100

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